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Diagonalizing the Frobenius

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Abstract

Over a Noetherian, local ring R of prime characteristic p, the Frobenius functor F_R induces a diagonalizable map on certain quotients of rational Grothendieck groups. This leads to an explicit formula for the Dutta multiplicity, and it is shown that a weaker version of Serre's vanishing conjecture holds whenever $\chi(F_R(X)) = p^{\dim R}\chi(X)$ for all bounded complexes X of finitely generated, projective modules with finite length homology.

Keywords: Grothendieck space; Frobenius; Vanishing; Vanishing dimension; Intersection multiplicity; Dutta multiplicity

1. Introduction

For finitely generated modules M and N over a commutative, Noetherian, local ring R with $\operatorname{pd} M < \infty$ and $\ell(M \otimes_R N) < \infty$, the *intersection multiplicity* defined by Serre [12] is given by

$$\chi(M,N) = \sum_i (-1)^i \, \ell\bigl(\operatorname{Tor}_i^R(M,N)\bigr).$$

The *vanishing conjecture*, also formulated by Serre, states that

$$\chi(M,N) = 0 \quad \text{whenever } \dim M + \dim N < \dim R.$$

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Serre's original conjecture requires R to be regular, but the conjecture makes sense in the more general setting presented above. Serre proved that the vanishing conjecture holds when R is regular and of equal characteristic or unramified of mixed characteristic. Roberts [9] and, independently, Gillet and Soulé [5] later proved the conjecture in the more general setting where the requirement that R be regular is weakened to the requirement that R be a complete intersection and both modules have finite projective dimension. Foxby [3] proved that the conjecture generally holds when $\dim N \leqslant 1$.

However, the vanishing conjecture does not hold in the full generality presented above. This was shown in the famous counterexample by Dutta, Hochster and McLaughlin [2]. Subsequently, other counterexamples have emerged, such as the one by Miller and Singh [7].

For rings with prime characteristic p, a different intersection multiplicity was introduced by Dutta [1]. The *Dutta multiplicity* is given when $\dim M + \dim N \leqslant \dim R$ by

$$\chi_\infty(M,N) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} M}} \chi\big(F_R^e(M), N\big),$$

where F_R denotes the Frobenius functor. The Dutta multiplicity satisfies the vanishing conjecture and is equal to the usual intersection multiplicity whenever this satisfies the vanishing conjecture.

This paper studies the interplay between the vanishing conjecture and the Frobenius functor. The investigations are performed by studying *Grothendieck spaces* which are tensor products of $\mathbb{Q}$ with homomorphic images of Grothendieck groups of complexes. Proposition 11 shows that the class of a bounded complex of finitely generated, projective modules in a Grothendieck space satisfies the vanishing conjecture if and only if the Frobenius functor acts on it by multiplication by a constant. Following this is Theorem 12, which describes how to decompose such a class of a complex into eigenvectors for the Frobenius. This leads in Remark 14 to the following formula for the Dutta multiplicity:

$$\chi_\infty(M,N) = (1 \quad 0 \quad \cdots \quad 0) \begin{pmatrix} 1 & 1 & \cdots & 1 \\ p^t & p^{t-1} & \cdots & p^{t-u} \\ \vdots & \vdots & \ddots & \vdots \\ p^{ut} & p^{u(t-1)} & \cdots & p^{u(t-u)} \end{pmatrix}^{-1} \begin{pmatrix} \chi(M,N) \\ \chi(F_R(M),N) \\ \vdots \\ \chi(F_R^u(M),N) \end{pmatrix}.$$

Here, t is the co-dimension of M and u is a number that, in a sense, measures how far M is from satisfying the vanishing conjecture. The formula can be useful, for example when using a computer to calculate Dutta multiplicity. It should be noted that the diagonalizability of the Frobenius functor has been discussed by Kurano [6], but that the approach taken and the results obtained in this paper are new, at least to the knowledge of this author.

The last section of this paper introduces the concept of *numerical vanishing*, a condition which holds if the vanishing conjecture holds, and which implies a weaker version of the vanishing conjecture, namely the one in which both modules are required to have finite projective dimension. A consequence of the investigations performed is the result from Remark 22 that the weak vanishing conjecture holds whenever $\chi(F_R(X)) = p^{\dim R}\chi(X)$ for all bounded complexes X of finitely generated, projective modules with finite length homology.

2. Notation

Throughout this paper, R denotes a commutative, Noetherian, local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Modules and complexes are, unless otherwise stated, assumed to be R-modules and R-complexes, respectively. Modules are considered to be complexes concentrated in degree zero.

The *spectrum* of R, denoted $\operatorname{Spec} R$, is the set of prime ideals of R. A subset $\mathfrak{X} \subseteq \operatorname{Spec} R$ is *specialization-closed* if, for any inclusion $\mathfrak{p} \subseteq \mathfrak{q}$ of prime ideals, $\mathfrak{p} \in \mathfrak{X}$ implies $\mathfrak{q} \in \mathfrak{X}$. A closed subset of $\operatorname{Spec} R$ is, in particular, specialization-closed. Throughout, whenever we deal with subsets of the spectrum of a ring, it is implicitly assumed that they are non-empty and specialization-closed.

For every $\mathfrak{X} \subseteq \operatorname{Spec} R$, the *dimension* of $\mathfrak{X}$, denoted $\dim \mathfrak{X}$, is the usual Krull dimension of $\mathfrak{X}$, and the *co-dimension* of $\mathfrak{X}$, denoted $\operatorname{codim} \mathfrak{X}$, is the number $\dim R - \dim \mathfrak{X}$. The dimension and co-dimension of a complex X (and hence also of a module) is the dimension and co-dimension of its support: that is, of the set $\operatorname{Supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(X_{\mathfrak{p}}) \neq 0\}$.

3. Grothendieck spaces and vanishing

For every (non-empty, specialization-closed) $\mathfrak{X} \subseteq \operatorname{Spec} R$, consider the following categories:

$\mathsf{P}(\mathfrak{X})$ = the category of bounded complexes with support contained in $\mathfrak{X}$ and consisting of finitely generated, projective modules,

$\mathsf{C}(\mathfrak{X})$ = the category of homologically bounded complexes with support contained in $\mathfrak{X}$ and with finitely generated homology modules.

If $\mathfrak{X} = \{\mathfrak{m}\}$, we simply write $\mathsf{P}(\mathfrak{m})$ and $\mathsf{C}(\mathfrak{m})$.

The *Euler characteristic* of a complex X in $\mathsf{C}(\mathfrak{m})$ is the integer

$$\chi(X) = \sum_i (-1)^i \,\ell\big(\mathrm{H}_i(X)\big).$$

If M and N are finitely generated modules with $\operatorname{pd} M < \infty$ and $\ell(M \otimes_R N) < \infty$, and X is a projective resolution of M, $X \otimes_R N$ is a complex in $\mathsf{C}(\mathfrak{m})$, and the intersection multiplicity $\chi(M, N)$ of M and N is the number $\chi(X \otimes_R N)$. There is no problem in letting N be a complex rather than just a module, so the definition of intersection multiplicity can be extended to an even more general setting: for subsets $\mathfrak{X}, \mathfrak{Y} \subseteq \operatorname{Spec} R$ with $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and complexes $X \in \mathsf{P}(\mathfrak{X})$ and $Y \in \mathsf{C}(\mathfrak{Y})$, the intersection multiplicity of X and Y is defined as

$$\chi(X, Y) = \chi(X \otimes_R Y) = \sum_i (-1)^i \,\ell\big(\mathrm{H}_i(X \otimes_R Y)\big).$$

In the construction of Grothendieck spaces below, the extra requirement that $\dim \mathfrak{X} + \dim \mathfrak{Y} \leqslant \dim R$ is needed; this corresponds to the assumption that $\dim M + \dim N \leqslant \dim R$, which is necessary in order to define the Dutta multiplicity. To formalize this, define, for each $\mathfrak{X} \subseteq \operatorname{Spec} R$, the subset

$$\mathfrak{X}^c = \big\{\mathfrak{q} \in \operatorname{Spec} R \;\big|\; \mathfrak{X} \cap V(\mathfrak{q}) = \{\mathfrak{m}\} \text{ and } \dim V(\mathfrak{q}) \leqslant \operatorname{codim} \mathfrak{X}\big\}.$$

The set $\mathfrak{X}^c$ is the largest specialization-closed subset of $\operatorname{Spec} R$ such that

$$\mathfrak{X} \cap \mathfrak{X}^c = \{\mathfrak{m}\} \quad \text{and} \quad \dim \mathfrak{X} + \dim \mathfrak{X}^c \leqslant \dim R.$$

(It is not hard to see that, when $\mathfrak{X}$ is closed, $\dim \mathfrak{X} + \dim \mathfrak{X}^c = \dim R$.) Thus, for $\mathfrak{X}, \mathfrak{Y} \subseteq \operatorname{Spec} R$, the property that $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{Y} \leqslant \dim R$ is equivalent to $\mathfrak{Y} \subseteq \mathfrak{X}^c$ which again is equivalent to $\mathfrak{X} \subseteq \mathfrak{Y}^c$.

Definition 1. Let $\mathfrak{X} \subseteq \operatorname{Spec} R$. The *Grothendieck space* of the category $\mathsf{P}(\mathfrak{X})$ is the $\mathbb{Q}$-vector space $\mathbb{G}\mathsf{P}(\mathfrak{X})$ presented by elements $[X]$, one for each isomorphism class of a complex X in $\mathsf{P}(\mathfrak{X})$, and relations

$$[X] = [\widetilde{X}] \quad \text{whenever } \chi(X, -) = \chi(\widetilde{X}, -) : \mathsf{C}\big(\mathfrak{X}^c\big) \to \mathbb{Q}.$$

Similarly, the *Grothendieck space* of the category $\mathsf{C}(\mathfrak{X})$ is the $\mathbb{Q}$-vector space $\mathbb{G}\mathsf{C}(\mathfrak{X})$ presented by elements $[Y]$, one for each isomorphism class of a complex Y in $\mathsf{C}(\mathfrak{X})$, and relations

$$[Y] = [\widetilde{Y}] \quad \text{whenever } \chi(-, Y) = \chi(-, \widetilde{Y}) : \mathsf{P}\big(\mathfrak{X}^c\big) \to \mathbb{Q}.$$

If $\mathfrak{X} = \{\mathfrak{m}\}$, we simply write $\mathbb{G}\mathsf{P}(\mathfrak{m})$ and $\mathbb{G}\mathsf{C}(\mathfrak{m})$.

Since intersection multiplicity is additive on short exact sequences and trivial on exact complexes, the Grothendieck spaces $\mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\mathbb{G}\mathsf{C}(\mathfrak{X})$ can also be regarded as the tensor product of $\mathbb{Q}$ with quotients of the Grothendieck groups $K_0(\mathsf{P}(\mathfrak{X}))$ and $K_0(\mathsf{C}(\mathfrak{X}))$ of the categories $\mathsf{P}(\mathfrak{X})$ and $\mathsf{C}(\mathfrak{X})$. (For further details on Grothendieck groups of categories of complexes, see [4].) In particular, any relation in one of these Grothendieck groups is also a relation in the corresponding Grothendieck space.

Intersection multiplicity in one variable naturally induces $\mathbb{Q}$-linear maps

$$\chi(-, Y) : \mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by } \chi\big([X], Y\big) = \chi(X, Y)$$

for each $Y \in \mathsf{C}(\mathfrak{X}^c)$. We equip $\mathbb{G}\mathsf{P}(\mathfrak{X})$ with the initial topology of these maps: this is the coarsest topology such that all the maps are continuous. Likewise, there are naturally induced $\mathbb{Q}$-linear maps

$$\chi(X, -) : \mathbb{G}\mathsf{C}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by } \chi\big(X, [Y]\big) = \chi(X, Y)$$

for each $X \in \mathsf{P}(\mathfrak{X}^c)$, and we equip $\mathbb{G}\mathsf{C}(\mathfrak{X})$ with the initial topology of these maps. It is straightforward to see that addition and scalar multiplication are continuous operations, making $\mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\mathbb{G}\mathsf{C}(\mathfrak{X})$ topological $\mathbb{Q}$-vector spaces. Henceforth, Grothendieck spaces are always considered to be topological $\mathbb{Q}$-vector spaces, so that, for example, a "homomorphism" between Grothendieck spaces is a continuous and $\mathbb{Q}$-linear map.

Proposition 2. *Suppose that* $\mathfrak{X}, \mathfrak{Y} \subseteq \operatorname{Spec} R$.

(i) *If* $0 \to X \to Y \to Z \to 0$ *is a short exact sequence of complexes in* $\mathsf{P}(\mathfrak{X})$ (*or in* $\mathsf{C}(\mathfrak{X})$, *respectively*), *then* $[Y] = [X] + [Z]$ *in* $\mathbb{G}\mathsf{P}(\mathfrak{X})$ (*or in* $\mathbb{G}\mathsf{C}(\mathfrak{X})$, *respectively*).

(ii) *If* $\varphi : X \to Y$ *is a quasi-isomorphism of complexes in* $\mathsf{P}(\mathfrak{X})$ (*or in* $\mathsf{C}(\mathfrak{X})$*, respectively*)*, then* $[X] = [Y]$ *in* $\mathbb{G}\mathsf{P}(\mathfrak{X})$ (*or in* $\mathbb{G}\mathsf{C}(\mathfrak{X})$*, respectively*)*. In particular, if* X *is exact, then* $[X] = 0$.

(iii) *If* X *is a complex in* $\mathsf{P}(\mathfrak{X})$ (*or in* $\mathsf{C}(\mathfrak{X})$*, respectively*)*, then* $[\Sigma^n X] = (-1)^n[X]$ *in* $\mathbb{G}\mathsf{P}(\mathfrak{X})$ (*or in mathbbG*$\mathsf{C}(\mathfrak{X})$*, respectively*)*.* (*Here,* $\Sigma^n(-)$ *denotes the shift functor, taking a complex* X *to the complex* $\Sigma^n X$ *defined by* $(\Sigma^n X)_i = X_{i-n}$ *and* $\partial_i^{\Sigma^n X} = (-1)^n \partial_{i-n}^X$.)

(iv) *Any element in* $\mathbb{G}\mathsf{P}(\mathfrak{X})$ (*or in* $\mathbb{G}\mathsf{C}(\mathfrak{X})$*, respectively*) *can be written in the form* $r[X]$ *for a rational number* $r \in \mathbb{Q}$ *and a complex* X *in* $\mathsf{P}(\mathfrak{X})$ (*or in* $\mathsf{C}(\mathfrak{X})$*, respectively*)*.*

(v) $\mathbb{G}\mathsf{C}(\mathfrak{X})$ *is generated by the elements* $[R/\mathfrak{q}]$ *for prime ideals* $\mathfrak{q} \in \mathfrak{X}$.

(vi) *The Euler characteristic* $\chi : \mathsf{C}(\mathfrak{m}) \to \mathbb{Q}$ *induces an isomorphism* (*that is, a* $\mathbb{Q}$*-linear homeomorphism*)

$$\chi : \mathbb{G}\mathsf{C}(\mathfrak{m}) \xrightarrow{\cong} \mathbb{Q} \quad \text{given by } \chi\big([X]\big) = \chi(X).$$

(vii) *The inclusion* $\mathsf{P}(\mathfrak{X}) \to \mathsf{C}(\mathfrak{X})$ *and, when* $\mathfrak{X} \subseteq \mathfrak{Y}$*, the inclusions* $\mathsf{P}(\mathfrak{X}) \to \mathsf{P}(\mathfrak{Y})$ *and* $\mathsf{C}(\mathfrak{X}) \to \mathsf{C}(\mathfrak{Y})$ *of categories induce homomorphisms* $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{C}(\mathfrak{X})$, $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}(\mathfrak{Y})$ *and* $\mathbb{G}\mathsf{C}(\mathfrak{X}) \to \mathbb{G}\mathsf{C}(\mathfrak{Y})$ *given in all cases by* $[X] \mapsto [X]$.

(viii) *If* $\mathfrak{Y} \subseteq \mathfrak{X}^c$*, the tensor product of complexes induces bi-homomorphisms* (*homomorphisms in each variable*)

$$- \otimes - : \mathbb{G}\mathsf{P}(\mathfrak{X}) \times \mathbb{G}\mathsf{C}(\mathfrak{Y}) \to \mathbb{G}\mathsf{C}(\mathfrak{m}) \quad \text{and}$$
$$- \otimes - : \mathbb{G}\mathsf{P}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}(\mathfrak{Y}) \to \mathbb{G}\mathsf{P}(\mathfrak{m})$$

given in both cases by $[X] \otimes [Y] = [X \otimes_R Y]$.

Proof. Properties (i), (ii) and (iii) hold since they hold for the corresponding Grothendieck groups; see [4].

We show that (iv) holds for elements in $\mathbb{G}\mathsf{P}(\mathfrak{X})$; the argument for elements in $\mathbb{G}\mathsf{C}(\mathfrak{X})$ is identical. Note first that any element in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ can be written as a sum $\sum_i r_i[X^i]$ for various complexes X^i in $\mathsf{P}(\mathfrak{X})$. By using (iii), we can assume that all r_i are positive, and by choosing a greatest common divisor, we can write the element in the form $r\sum_i a_i[X^i]$ for a rational number r and positive integers a_i. Because of (i), a sum of two elements represented by complexes is equal to the element represented by their direct sum, and hence the sum $\sum_i a_i[X^i]$ can be replaced by a single element $[X]$, where X is the direct sum over i of a_i copies of X^i.

Property (v) holds since it holds for the corresponding Grothendieck group. This is easily seen by using short exact sequences to transform a complex in $\mathsf{C}(\mathfrak{X})$ first into a bounded complex, then into the alternating sum of its homology modules, and finally, by taking filtrations, into a linear combination of modules in the form $R/\mathfrak{q}$ for prime ideals $\mathfrak{q} \in \mathfrak{X}$.

The $\mathbb{Q}$-vector space isomorphism in (vi) is an immediate consequence of the group isomorphism $K_0(\mathsf{C}(\mathfrak{m})) \xrightarrow{\cong} \mathbb{Z}$ induced by the Euler characteristic on Grothendieck groups. It is straightforward to see that it is a homeomorphism.

To see (vii), it suffices to note that, since $\mathsf{C}(\mathfrak{X}^c)$ contains $\mathsf{P}(\mathfrak{X}^c)$ as well as $\mathsf{C}(\mathfrak{Y}^c)$ whenever $\mathfrak{X} \subseteq \mathfrak{Y}$ (because then $\mathfrak{Y}^c \subseteq \mathfrak{X}^c$), any relation in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ is also a relation in $\mathbb{G}\mathsf{C}(\mathfrak{X})$ and $\mathbb{G}\mathsf{P}(\mathfrak{Y})$.

Finally, (viii) simply follows from the definition of Grothendieck spaces. As an example, we show that the second map in (viii) is a homomorphism in the first variable. So fix $Y \in \mathsf{P}(\mathfrak{Y})$ and let $Z \in \mathsf{C}(\{\mathfrak{m}\}^c) = \mathsf{C}(\operatorname{Spec} R)$ be arbitrary. Then

$$\chi(- \otimes_R Y, Z) = \chi(-, Y \otimes_R Z) : \mathsf{P}(\mathfrak{X}) \to \mathbb{Q},$$

which shows that the map $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}(\mathfrak{m})$ given by $[X] \mapsto [X \otimes_R Y]$ is well-defined, $\mathbb{Q}$-linear and continuous. $\square$

The homomorphisms in Proposition 2(vii) are called *inclusion homomorphisms* although they in general are not injective. The image under an inclusion homomorphism of an element α will generally be denoted $\bar{\alpha}$.

Let $\mathfrak{X}, \mathfrak{Y} \subseteq \operatorname{Spec} R$ with $\mathfrak{Y} \subseteq \mathfrak{X}^c$ and suppose that $X \in \mathsf{P}(\mathfrak{X})$ and $Y \in \mathsf{C}(\mathfrak{Y})$. Then

$$\chi(X, Y) = \chi(X \otimes_R Y) = \chi\big([X \otimes_R Y]\big) = \chi\big([X] \otimes [Y]\big),$$

which is the image in $\mathbb{Q}$ of $[X] \otimes [Y]$ under the isomorphism $\mathbb{G}\mathsf{C}(\mathfrak{m}) \cong \mathbb{Q}$ induced by the Euler characteristic. Thus, the intersection multiplicity of complexes generalizes to the bi-homomorphism $\mathbb{G}\mathsf{P}(\mathfrak{X}) \times \mathbb{G}\mathsf{C}(\mathfrak{Y}) \to \mathbb{G}\mathsf{C}(\mathfrak{m})$ from Proposition 2(viii).

Definition 3. Given $\mathfrak{X} \subseteq \operatorname{Spec} R$ and elements $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X})$, the *dimensions* of α and β are defined as

$$\dim \alpha = \inf\big\{\dim X \;\big|\; \alpha = r[X] \text{ for some } r \in \mathbb{Q} \text{ and } X \in \mathsf{P}(\mathfrak{X})\big\} \quad \text{and}$$
$$\dim \beta = \inf\big\{\dim Y \;\big|\; \beta = s[Y] \text{ for some } s \in \mathbb{Q} \text{ and } Y \in \mathsf{C}(\mathfrak{X})\big\}.$$

In particular, $\dim \alpha = -\infty$ if and only if $\alpha = 0$.

Definition 4. Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. Then α *satisfies vanishing* if, for all $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$, $\alpha \otimes \beta = 0$ whenever $\dim \beta < \operatorname{codim} \mathfrak{X}$, and α satisfies *weak vanishing* if, for all $\beta \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)$, $\overline{\alpha \otimes \beta} = 0$ in $\mathbb{G}\mathsf{C}(\mathfrak{m})$ whenever $\dim \beta < \operatorname{codim} \mathfrak{X}$. The *vanishing dimension* of α is the number

$$\operatorname{vdim} \alpha = \inf\big\{u \in \mathbb{Z} \;\big|\; \alpha \otimes \beta = 0 \text{ for all } \beta \in \mathbb{G}\mathsf{C}\big(\mathfrak{X}^c\big) \text{ with } \dim \beta < \operatorname{codim} \mathfrak{X} - u\big\}.$$

In particular, $\operatorname{vdim} \alpha = -\infty$ if and only if $\alpha = 0$, and $\operatorname{vdim} \alpha \leqslant 0$ if and only if α satisfies vanishing.

To satisfy vanishing and weak vanishing for an element α generalizes the usual terminology for complexes: if $X \in \mathsf{P}(\mathfrak{X})$, then the element $[X]$ in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ satisfies vanishing exactly when $\chi(X, Y) = 0$ for all $Y \in \mathsf{C}(\mathfrak{X}^c)$. Likewise, $[X]$ satisfies weak vanishing exactly when $\chi(X, Y) = 0$ for all $Y \in \mathsf{P}(\mathfrak{X}^c)$.

The vanishing dimension measures, in a sense, how far an element is from satisfying vanishing: if $\operatorname{vdim}[X] = u$, then u is the smallest integer such that $\chi(X, Y) = 0$ for all $Y \in \mathsf{C}(\mathfrak{X}^c)$ with $\dim X + \dim Y < \dim R - u$.

Remark 5. A result by Foxby [3] shows that vanishing holds for all $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ whenever $\operatorname{codim} \mathfrak{X} \leqslant 2$. In particular, for all $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$,

$$\operatorname{vdim} \alpha \leqslant \max(0, \operatorname{codim} \mathfrak{X} - 2).$$

Proposition 6. *Suppose that* $\mathfrak{X} \subseteq \operatorname{Spec} R$*, let* $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ *and let* u *be a non-negative integer. The following are equivalent.*

(i) $\alpha \otimes \beta = 0$ *for all* $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ *with* $\dim \beta < \operatorname{codim} \mathfrak{X} - u$.
(ii) $\bar{\alpha}$ *satisfies vanishing in* $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ *for all* $\mathfrak{Y} \supseteq \mathfrak{X}$ *with* $\operatorname{codim} \mathfrak{Y} = \operatorname{codim} \mathfrak{X} - u$.
(iii) $\bar{\alpha} = 0$ *in* $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ *for all* $\mathfrak{Y} \supseteq \mathfrak{X}$ *with* $\operatorname{codim} \mathfrak{Y} < \operatorname{codim} \mathfrak{X} - u$.
(iv) $\bar{\alpha} = 0$ *in* $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ *for all* $\mathfrak{Y} \supseteq \mathfrak{X}$ *with* $\operatorname{codim} \mathfrak{Y} = \operatorname{codim} \mathfrak{X} - u - 1$.
(v) $\operatorname{vdim} \alpha \leqslant u$.

Proof. Straightforward. □

Remark 7. Suppose that $\mathfrak{X} \subseteq \mathfrak{Y}$, let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and denote by $\bar{\alpha}$ the image in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ of α under the inclusion homomorphism. Then

$$\operatorname{vdim} \bar{\alpha} \leqslant \operatorname{vdim} \alpha - (\operatorname{codim} \mathfrak{X} - \operatorname{codim} \mathfrak{Y}).$$

It is always possible to find a $\mathfrak{Y} \supseteq \mathfrak{X}$ with any given co-dimension larger than or equal to $\operatorname{codim} \mathfrak{X} - \operatorname{vdim} \alpha$ and smaller than or equal to $\operatorname{codim} \mathfrak{X}$ such that the above is an equality.

4. Frobenius and vanishing dimension

Assumption. Throughout this section, R is assumed to be complete of prime characteristic p, and k is assumed to be a perfect field.[2]

The Frobenius ring homomorphism $f : R \to R$ is given by $f(r) = r^p$; the e-fold composition of f is the ring homomorphism $f^e : R \to R$ given by $f(r) = r^{p^e}$. We denote ${}^{f^e}R$ the bi-R-algebra R having the structure of an R-algebra from the left by f^e and from the right by the identity map: that is, if $x \in {}^{f^e}R$ and $r, s \in R$, then $r \cdot x \cdot s = r^{p^e} xs$.

Definition 8. Two functors, ${}^{f^e}(-)$ and F_R^e, are defined on the category of R-modules by

$${}^{f^e}(-) = {}^{f^e}R \otimes_R - \quad \text{and} \quad F_R^e(-) = - \otimes_R {}^{f^e}R,$$

where, for a module M, ${}^{f^e}M$ is viewed through its *left* structure, whereas $F_R^e(M)$ is viewed through its *right* structure. The functor F_R is called the *Frobenius functor*.

Like the usual intersection multiplicity, the definition of Dutta multiplicity can be extended to a more general setting: for subsets $\mathfrak{X}, \mathfrak{Y} \subseteq \operatorname{Spec} R$ with $\mathfrak{Y} \subseteq \mathfrak{X}^c$ and complexes $X \in \mathsf{P}(\mathfrak{X})$ and $Y \in \mathsf{C}(\mathfrak{Y})$, the Dutta multiplicity of X and Y is defined as

$$\chi_\infty(X, Y) = \lim_{e \to \infty} \frac{1}{p^{e \operatorname{codim} X}} \chi\big(F_R^e(X), Y\big).$$

Proposition 9. *The following hold.*

(i) *For all* $\mathfrak{X} \subseteq \operatorname{Spec} R$, ${}^{f^e}(-)$ *defines an exact functor* $\mathsf{C}(\mathfrak{X}) \to \mathsf{C}(\mathfrak{X})$.
(ii) *For all* $\mathfrak{X} \subseteq \operatorname{Spec} R$, F_R *defines a functor* $\mathsf{P}(\mathfrak{X}) \to \mathsf{P}(\mathfrak{X})$.
(iii) ${}^{f^e}(-)$ *and* F_R^e *are the compositions of e copies of* ${}^{f}(-)$ *and* F_R, *respectively.*

[2] Note that, although the assumptions that R be complete and k be perfect may seem restrictive, they really are not when it comes to dealing with intersection multiplicities; for further details, see Dutta [1, p. 425].

Proof. All properties are readily verified. For further details, see, for example, Peskine and Szpiro [8] or Roberts [11]. □

According to Proposition 9(i), for any complex $Z \in \mathsf{C}(\mathfrak{m})$,

$$\chi\big({}^{f^e}Z\big) = \chi(Z)\,\ell\big({}^{f^e}k\big) = \chi(Z),$$

where the last equation follows since $k \cong {}^{f^e}k$. Now, suppose that $X \in \mathsf{P}(\mathfrak{X})$ and $Y \in \mathsf{C}(\mathfrak{X}^c)$. It is not hard to see that ${}^{f^e}(F_R^e(X) \otimes_R Y) \cong X \otimes_R {}^{f^e}Y$, and it follows that

$$\chi\big(F_R^e(-), Y\big) = \chi\big(-, {}^{f^e}Y\big) : \mathsf{P}(\mathfrak{X}) \to \mathbb{Q}, \tag{1}$$

which implies that the map $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}(\mathfrak{X})$ given by $[X] \mapsto [F_R^e(X)]$ is well-defined, $\mathbb{Q}$-linear and continuous; in other words, it is an endomorphism of Grothendieck spaces.

Definition 10. Given $\mathfrak{X} \subseteq \operatorname{Spec} R$ and $e \in \mathbb{N}_0$, the endomorphism on $\mathbb{G}\mathsf{P}(\mathfrak{X})$ induced by F_R^e is denoted $F_{\mathfrak{X}}^e$. Further, we define the endomorphism

$$\Phi_{\mathfrak{X}}^e = \frac{1}{p^{e \operatorname{codim} \mathfrak{X}}} F_{\mathfrak{X}}^e$$

on $\mathbb{G}\mathsf{P}(\mathfrak{X})$. For $\mathfrak{X} = \{\mathfrak{m}\}$ we simply write $F_{\mathfrak{m}}^e$ and $\Phi_{\mathfrak{m}}^e$.

Proposition 11. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. Then α satisfies vanishing if and only if $\alpha = \Phi_{\mathfrak{X}}(\alpha)$.*

Proof. According to Proposition 2(iv), we can assume that α is in the form $\alpha = r[X]$ for $r \in \mathbb{Q}$ and $X \in \mathsf{P}(\mathfrak{X})$. By Proposition 2(v) and the definition of Grothendieck spaces, the element α is completely determined by the intersection multiplicities $\chi(\alpha, R/\mathfrak{q})$ for prime ideals $\mathfrak{q} \in \mathfrak{X}^c$. Given such a prime ideal $\mathfrak{q}$, set $m = \dim R/\mathfrak{q}$ and note that, since $R/\mathfrak{q}$ is a complete domain of characteristic p and with perfect residue field, $R/\mathfrak{q}$ is torsion-free of rank p^m over ${}^f(R/\mathfrak{q})$; see Roberts [11, Section 7.3]. Thus, there is a short exact sequence

$$0 \to (R/\mathfrak{q})^{p^m} \to {}^f(R/\mathfrak{q}) \to Q \to 0,$$

where Q is a finitely generated module with $\dim Q < m$. By applying (1), we get

$$\chi\big(F_R(X), R/\mathfrak{q}\big) = p^m \chi(X, R/\mathfrak{q}) + \chi(X, Q).$$

Setting $t = \operatorname{codim} \mathfrak{X} \geqslant m$, this means that

$$\chi\big(\Phi_{\mathfrak{X}}(\alpha), R/\mathfrak{q}\big) = p^{m-t}\chi(\alpha, R/\mathfrak{q}) + p^{-t}\chi(\alpha, Q). \tag{2}$$

Now, if α satisfies vanishing, formula (2) shows that α and $\Phi_{\mathfrak{X}}(\alpha)$ yield the same intersection multiplicities with $R/\mathfrak{q}$ for all $\mathfrak{q} \in \mathfrak{X}^c$, which means that $\alpha = \Phi_{\mathfrak{X}}(\alpha)$. Conversely, if $\alpha = \Phi_{\mathfrak{X}}(\alpha)$, then formula (2) implies that

$$\big(p^t - p^m\big)\chi(\alpha, R/\mathfrak{q}) = \chi(\alpha, Q),$$

which means that α satisfies vanishing: for if this were not the case, one could choose $\mathfrak{q} \in \mathfrak{X}^c$ with $m = \dim R/\mathfrak{q} < t$ minimal such that $\chi(\alpha, R/\mathfrak{q}) \neq 0$, and minimality of m would then imply that $\chi(\alpha, Q) = 0$ which gives a contradiction. $\square$

Theorem 12. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$, let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and suppose that u is a non-negative integer with $u \geqslant \operatorname{vdim} \alpha$. Then*

$$\big(p^u \Phi_{\mathfrak{X}} - \mathrm{id}\big) \circ \cdots \circ (p\Phi_{\mathfrak{X}} - \mathrm{id}) \circ (\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha) = 0. \tag{3}$$

Further, there exists a decomposition $\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$ in which each $\alpha^{(i)}$ is either zero or an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue $1/p^i$. The elements $\alpha^{(0)}, \ldots, \alpha^{(u)}$ can be recursively defined by

$$\alpha^{(0)} = \lim_{e\to\infty} \Phi_{\mathfrak{X}}^e(\alpha) \quad \textit{and} \quad \alpha^{(i)} = \lim_{e\to\infty} p^{ie}\Phi_{\mathfrak{X}}^e\big(\alpha - \big(\alpha^{(0)} + \cdots + \alpha^{(i-1)}\big)\big),$$

and there is a formula

$$\begin{pmatrix} \alpha^{(0)} \\ \vdots \\ \alpha^{(u)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1/p & \cdots & 1/p^u \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1/p^u & \cdots & 1/p^{u^2} \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \Phi_{\mathfrak{X}}(\alpha) \\ \vdots \\ \Phi_{\mathfrak{X}}^u(\alpha) \end{pmatrix}. \tag{4}$$

Proof. We prove (3) by induction on u. The case $u = 0$ is trivial since Proposition 11 in this situation yields that $(\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha) = 0$. Now, suppose that $u > 0$ and that the formula holds for smaller values of u. By Proposition 11 and commutativity of the involved maps, Eq. (3) holds if and only if vanishing holds for the element

$$\beta = \big(p^u \Phi_{\mathfrak{X}} - \mathrm{id}\big) \circ \cdots \circ (p\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha).$$

Now, let $\mathfrak{Y} \subseteq \operatorname{Spec} R$ with $\mathfrak{Y} \supseteq \mathfrak{X}$ and $\operatorname{codim} \mathfrak{Y} = \operatorname{codim} \mathfrak{X} - 1$. Then, in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$, $\overline{\Phi_{\mathfrak{X}}(\alpha)} = p^{-1}\Phi_{\mathfrak{Y}}(\bar{\alpha})$, and hence

$$\bar{\beta} = \big(p^{u-1}\Phi_{\mathfrak{Y}} - \mathrm{id}\big) \circ \cdots \circ (p\Phi_{\mathfrak{Y}} - \mathrm{id}) \circ (\Phi_{\mathfrak{Y}} - \mathrm{id})(\bar{\alpha}) = 0,$$

where the last equation follows by induction, since $\operatorname{vdim} \bar{\alpha} \leqslant u - 1$ by Remark 7. According to Proposition 6, this proves that β satisfies vanishing.

By applying $\Phi_{\mathfrak{X}}^{e-u}$ to (3), we get a recursive formula to compute $\Phi_{\mathfrak{X}}^{e+1}(\alpha)$ from $\Phi_{\mathfrak{X}}^e(\alpha), \ldots, \Phi_{\mathfrak{X}}^{e-u}(\alpha)$. The characteristic polynomial for the recursion is

$$\big(p^u x - 1\big) \cdots (px - 1)(x - 1),$$

which has $u + 1$ distinct roots, namely $1, 1/p, \ldots, 1/p^u$. Thus, there is a general formula

$$\Phi_{\mathfrak{X}}^e(\alpha) = \alpha^{(0)} + \frac{1}{p^e}\alpha^{(1)} + \cdots + \frac{1}{p^{ue}}\alpha^{(u)} \tag{5}$$

for suitable $\alpha^{(0)}, \dots, \alpha^{(u)} \in \mathbb{G}\mathsf{P}(\mathfrak{X})$, where each $\alpha^{(i)}$ satisfies

$$\Phi_{\mathfrak{X}}^{e}\big(\alpha^{(i)}\big) = \frac{1}{p^{ei}}\alpha^{(i)} \tag{6}$$

and hence is an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue $1/p^{i}$.

We obtain the recursive definition of $\alpha^{(i)}$ by induction on i. The case $i = 0$ follows immediately from (5) by letting e go to infinity. Suppose now that $i > 0$ and that the result holds for smaller values of i. From (5) and (6) we then get

$$\begin{aligned} p^{ie}\Phi^{e}\big(\alpha - \big(\alpha^{(0)} + \cdots + \alpha^{(i-1)}\big)\big) &= p^{ie}\Phi^{e}\big(\alpha^{(i)} + \cdots + \alpha^{(u)}\big) \\ &= \alpha^{(i)} + \frac{1}{p^{e}}\alpha^{(i+1)} + \cdots + \frac{1}{p^{e(u-i)}}\alpha^{(u)}, \end{aligned}$$

and letting e go to infinity, we obtain the desired formula.

From (5) we know that $\alpha^{(0)}, \dots, \alpha^{(u)}$ solve the following system of equations with rational coefficients:

$$\begin{array}{ccccccccc} \alpha^{(0)} & + & \alpha^{(1)} & + & \cdots & + & \alpha^{(u)} & = & \alpha, \\ \alpha^{(0)} & + & \frac{1}{p}\alpha^{(1)} & + & \cdots & + & \frac{1}{p^{u}}\alpha^{(u)} & = & \Phi_{\mathfrak{X}}(\alpha), \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ \alpha^{(0)} & + & \frac{1}{p^{u}}\alpha^{(1)} & + & \cdots & + & \frac{1}{p^{u^2}}\alpha^{(u)} & = & \Phi_{\mathfrak{X}}^{u}(\alpha). \end{array}$$

Formula (4) now follows. (The matrix is the Vandermonde matrix of the elements $1, 1/p, \dots, 1/p^{u}$ with determinant $\prod_{0 \leqslant i < j \leqslant u}(1/p^{j} - 1/p^{i}) \neq 0$.) □

Remark 13. It is easy to see that, for $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\beta \in \mathbb{G}\mathsf{P}(\mathfrak{X}^{c})$,

$$(\alpha \otimes \beta)^{(t)} = \sum_{i+j=t}\alpha^{(i)} \otimes \beta^{(j)}$$

in $\mathbb{G}\mathsf{P}(\mathfrak{m})$. In particular, $(\alpha \otimes \beta)^{(0)} = \alpha^{(0)} \otimes \beta^{(0)}$. Suppose now that $\mathfrak{X} \subseteq \mathfrak{Y} \subseteq \operatorname{Spec} R$ and let $s = \operatorname{codim}\mathfrak{X} - \operatorname{codim}\mathfrak{Y}$. Since $\overline{\Phi_{\mathfrak{X}}(\alpha)} = p^{-s}\Phi_{\mathfrak{Y}}(\overline{\alpha})$ in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$, it follows from Theorem 12 that, in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$, $\overline{\alpha^{(i)}} = \overline{\alpha}^{(i-s)}$ for $i \geqslant s$ and $\overline{\alpha^{(i)}} = 0$ for $i < s$.

Remark 14. The Dutta multiplicity of an element $\alpha \in \mathsf{P}(\mathfrak{X})$ and complexes in $\mathsf{C}(\mathfrak{X}^{c})$ is given by applying the function

$$\chi_{\infty}(\alpha, -) = \lim_{e\to\infty}\chi\big(\Phi_{\mathfrak{X}}^{e}(\alpha), -\big) = \chi\Big(\lim_{e\to\infty}\Phi_{\mathfrak{X}}^{e}(\alpha), -\Big) = \chi\big(\alpha^{(0)}, -\big).$$

Thus, the Dutta multiplicity is a rational number and we need not find a limit to compute it. In fact, translating Theorem 12 back to the setup with complexes $X \in \mathsf{P}(\mathfrak{X})$ and $Y \in \mathsf{C}(\mathfrak{Y})$, where $\mathfrak{X} = \operatorname{Supp} X$, $\mathfrak{Y} = \operatorname{Supp} Y$ and $\mathfrak{Y} \subseteq \mathfrak{X}^{c}$, we obtain the general formula

$$\chi_\infty(X,Y) = (1 \quad 0 \quad \cdots \quad 0)\begin{pmatrix} 1 & 1 & \cdots & 1 \\ p^t & p^{t-1} & \cdots & p^{t-u} \\ \vdots & \vdots & \ddots & \vdots \\ p^{ut} & p^{u(t-1)} & \cdots & p^{u(t-u)} \end{pmatrix}^{-1} \begin{pmatrix} \chi(X,Y) \\ \chi(F_R(X),Y) \\ \vdots \\ \chi(F_R^u(X),Y) \end{pmatrix},$$

where $t = \operatorname{codim} X$ and $u \geqslant \operatorname{vdim}[X]$ for $[X] \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. The fact that Dutta multiplicity satisfies vanishing follows immediately from Proposition 15, which extends Proposition 6 by adding even more conditions that describe what it means to have a certain vanishing dimension.

Proposition 15. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$, let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and let u be a non-negative integer. The following are equivalent.*

(i) α *satisfies vanishing.*
(ii) $\alpha = \alpha^{(0)}$.
(iii) $\alpha = \Phi_{\mathfrak{X}}(\alpha)$.
(iv) $\alpha = \Phi_{\mathfrak{X}}^e(\alpha)$ *for some* $e \in \mathbb{N}$.
(v) $\alpha = \lim_{e\to\infty} \Phi_{\mathfrak{X}}^e(\alpha)$.

Further, the following are equivalent.

(vi) $\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$.
(vii) $(p^u\Phi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Phi_{\mathfrak{X}} - \mathrm{id}) \circ (\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha) = 0$.
(viii) $\operatorname{vdim} \alpha \leqslant u$.

Proof. (i) is equivalent to (iii) by Proposition 11; (iii) is equivalent to (ii) and (v) by Theorem 12; (iii) implies (iv) implies (v), so these must all be equivalent; the proof of Theorem 12 shows how (viii) implies (vii) which again implies (vi); and combining Remark 13 with Proposition 6 shows that (vi) implies (viii). □

Having vanishing dimension exactly equal to $u > 0$ of course means that conditions (vi)–(viii) are satisfied and that the same conditions fail to hold if u is replaced by $u - 1$. In particular, if $\operatorname{vdim} \alpha = u$, then $\alpha^{(u)} \neq 0$ and there exists a $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ with $\dim \beta = \operatorname{codim} \mathfrak{X} - u$ such that $\alpha \otimes \beta = \alpha^{(u)} \otimes \beta \neq 0$. Consequently, if the term $\alpha^{(i)}$ is non-zero, then it has vanishing dimension i and can be regarded as "the component of α that allows a counterexample to vanishing where the difference between co-dimension and dimension is equal to i."

5. Numerical vanishing

Assumption. Throughout this section, we continue to assume that R is complete of prime characteristic $p > 0$, and that k is a perfect field.

Definition 16. Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. We say that α satisfies *numerical vanishing* if $\bar{\alpha} = \overline{\alpha^{(0)}}$ in $\mathbb{G}\mathsf{C}(\mathfrak{X})$.

Proposition 17. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. For the following conditions, each condition implies the next.*

(i) α *satisfies vanishing.*
(ii) α *satisfies numerical vanishing.*
(iii) α *satisfies weak vanishing.*

Proof. It is clear from Proposition 15 that vanishing implies numerical vanishing. Suppose that α satisfies numerical vanishing and let $\beta \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)$ be such that $\dim \beta < \operatorname{codim} \mathfrak{X}$. Then

$$\overline{\alpha \otimes \beta} = \bar{\alpha} \otimes \beta = \overline{\alpha^{(0)}} \otimes \beta = \alpha^{(0)} \otimes \bar{\beta} = 0,$$

since $\alpha^{(0)}$ satisfies vanishing, and we conclude that α satisfies weak vanishing. $\square$

As Remark 22 will show, the implications in Proposition 17 are generally strict.

Remark 18. If X is a complex in $\mathsf{P}(\mathfrak{m})$, then, because of Proposition 2(vi), the element $[X] \in \mathbb{G}\mathsf{P}(\mathfrak{m})$ satisfies numerical vanishing if and only if

$$\lim_{e \to \infty} \frac{1}{p^{e \dim R}} \chi\left(F_R^e(X)\right) = \chi(X). \tag{7}$$

As we shall see in Proposition 19 below, for (7) to hold, it suffices (but need not be necessary) to verify that the equation

$$\chi\left(F_R^e(X)\right) = p^{e \dim R} \chi(X)$$

holds for $\operatorname{vdim}[X]$ distinct values of $e > 0$.

Proposition 19. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. A sufficient condition for α to satisfy numerical vanishing is that $\bar{\alpha} = \overline{\Phi_{\mathfrak{X}}^e(\alpha)}$ in $\mathbb{G}\mathsf{C}(\mathfrak{X})$ for* $\operatorname{vdim} \alpha$ *distinct values of $e > 0$.*

Proof. Let $u = \operatorname{vdim} \alpha$. According to Theorem 12, the difference $\overline{\Phi_{\mathfrak{X}}^e(\alpha)} - \bar{\alpha}$ in $\mathbb{G}\mathsf{C}(\mathfrak{X})$ is obtained by letting $x = 1/p^e$ in the polynomial

$$\left(\overline{\alpha^{(0)}} - \bar{\alpha}\right) + x\overline{\alpha^{(1)}} + \cdots + x^u \overline{\alpha^{(u)}}.$$

The polynomial always has the root $x = 1$. If there are u additional roots, it must be the zero-polynomial, so that $\bar{\alpha} = \overline{\alpha^{(0)}}$. $\square$

Definition 20. We say that R satisfies *vanishing* (or *numerical vanishing* or *weak vanishing*, respectively) if all elements of $\mathbb{G}\mathsf{P}(\mathfrak{X})$ satisfy vanishing (or numerical vanishing or weak vanishing, respectively) for all $\mathfrak{X} \subseteq \operatorname{Spec} R$.

Proposition 21. *The following are equivalent.*

(i) R *satisfies numerical vanishing.*
(ii) $\bar{\alpha} = \overline{\Phi_{\mathfrak{X}}(\alpha)}$ *in* $\mathbb{G}\mathsf{C}(\mathfrak{X})$ *for all* $\mathfrak{X} \subseteq \operatorname{Spec} R$ *and* $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$.
(iii) $\bar{\alpha} = \overline{\Phi_{\mathfrak{m}}(\alpha)}$ *in* $\mathbb{G}\mathsf{C}(\mathfrak{m})$ *for all* $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{m})$.
(iv) $\bar{\alpha} = \overline{\alpha^{(0)}}$ *in* $\mathbb{G}\mathsf{C}(\mathfrak{X})$ *for all* $\mathfrak{X} \subseteq \operatorname{Spec} R$ *and* $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$.
(v) $\bar{\alpha} = \overline{\alpha^{(0)}}$ *in* $\mathbb{G}\mathsf{C}(\mathfrak{m})$ *for all* $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{m})$.

Proof. By definition, (i) is equivalent to (iv). It is clear that (ii) implies (iii) and that (iv) implies (v). It is also clear that (ii) implies (iv) and that (iii) implies (v). Thus, it only remains to prove that (v) implies (ii). So assume (v) and let $\mathfrak{X} \subseteq \operatorname{Spec} R$ and $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. Then, for all $\beta \in \mathsf{P}(\mathfrak{X}^c)$,

$$\overline{\Phi_{\mathfrak{X}}(\alpha)} \otimes \beta = \overline{\Phi_{\mathfrak{X}}(\alpha) \otimes \beta} = \overline{\left(\Phi_{\mathfrak{X}}(\alpha) \otimes \beta\right)^{(0)}} = \overline{\Phi_{\mathfrak{X}}(\alpha)^{(0)} \otimes \beta^{(0)}} = \overline{\alpha^{(0)} \otimes \beta^{(0)}},$$

where we have applied Remark 13 and the fact that $\Phi_{\mathfrak{X}}(\alpha)^{(0)} = \alpha^{(0)}$. Similarly,

$$\bar{\alpha} \otimes \beta = \overline{\alpha \otimes \beta} = \overline{(\alpha \otimes \beta)^{(0)}} = \overline{\alpha^{(0)} \otimes \beta^{(0)}}.$$

Thus, $\bar{\alpha} = \overline{\Phi_{\mathfrak{X}}(\alpha)}$. $\square$

Remark 22. Comparing Remark 18 with Proposition 21, we see that a necessary and sufficient condition for R to satisfy numerical vanishing is that

$$\chi\left(F_R(X)\right) = p^{\dim R} \chi(X) \tag{8}$$

for all complexes $X \in \mathsf{P}(\mathfrak{m})$, and by Proposition 17, this condition implies that R satisfies weak vanishing.

Dutta [1] has proven that condition (8) holds when R is Gorenstein of dimension (at most) 3 or a complete intersection (of any dimension). The rings in the counterexamples by Dutta, Hochster and McLaughlin [2] and Miller and Singh [7] are complete intersections (which can be assumed to be complete of characteristic p and with perfect residue fields), and hence they satisfy numerical vanishing without satisfying vanishing.

Any ring of dimension at most 4 will satisfy weak vanishing; this follows from the result by Foxby [3]. Roberts [10] has shown the existence of a Cohen–Macaulay ring of dimension 3 (which can also be assumed to be complete of characteristic p and with perfect residue field) such that condition (8) does not hold. Thus, this ring satisfies weak vanishing without satisfying numerical vanishing.

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